Fracture Analysis of Bonded Elastomeric Disks Subjected to Triaxial Stress

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ABSTRACT: A theoretical model for the dependence of the modulus of elasticity, M, on the porosity, α, of an elastomeric disk subjected to triaxial stress was developed. An empirical law was proposed for the dependence of the surface energy, Γ, on the porosity, α, of the elastomer. The presented theoretical model is a two-degree-of-freedom model, the parameters of which are determined by fitting with experimental data derived from the elastomer disks. In order to obtain the final mathematical expression for the effective modules of the composite system, a relationship between the strain field within the disk and the porosity was developed. The numerical differentiation of the experimental stress/strain curve from the tested bonded elastomer disks yields the values of the *apparent* modulus of the elastomer disks as a function of the strain field within the testing specimens. An appropriate combination of the proposed theory and the received experimental data yields the percentage of the growing microvoids within the deformed material. © 1997 John Wiley & Sons, Inc. J Appl Polym Sci **65:** 1821–1827, 1997

Key words: elastomers; fracture analysis; porosity; apparent modulus; effective Poisson ratio

INTRODUCTION

In previous studies, experiments were performed on elastomeric disks subjected to triaxial stress. The so-called acoustic emission technique was applied for the evaluation and analysis of growing and tearing mechanisms of existing microvoids in the testing disks. It was shown[1,2] that the existing microvoids in the deformed disk are responsible for the reduction of the *apparent* modulus M, the stress softening, and the hysteresis when the material is subjected to triaxial stress conditions. A series of experiments conducted in our laboratory confirmed the existence of microvoids within the deformed, bonded, unfilled nitrite rubber disks. Valuable information about the size of the deformed voids in the material was obtained using the frequency spectrum of the detected acoustic emission signals.

The effects of voids on the response of a rubber pancake sample were examined in a previous experimental study by myself and coworkers.[3] Using the linear theory of stress analysis, a theoretical estimation of the *diametrical contraction* of the rubber pancake disks was obtained. Experimental measurements[3] of the *lateral contraction* at the middle plane of a pancake sample have shown that the testing rubber is no longer an incompressible material. By comparing the experimental data with the proposed theoretical equations, a value of 0.487 was assigned for the effective Poisson's ratio of the material. An *effective* Poisson's ratio v_{eff} was estimated in Reference 4, which is consistent with the measured values of 0.492 from the normalized *volumetric contraction* experimental data. A simple finite element mesh was used in order to obtain analytical expressions relating the volumetric contraction and $(M/E)_{tens.}$ to effective Poisson's ratio. Taking a typical experimental value of aspect ratio $(D/H = 10)$, shear

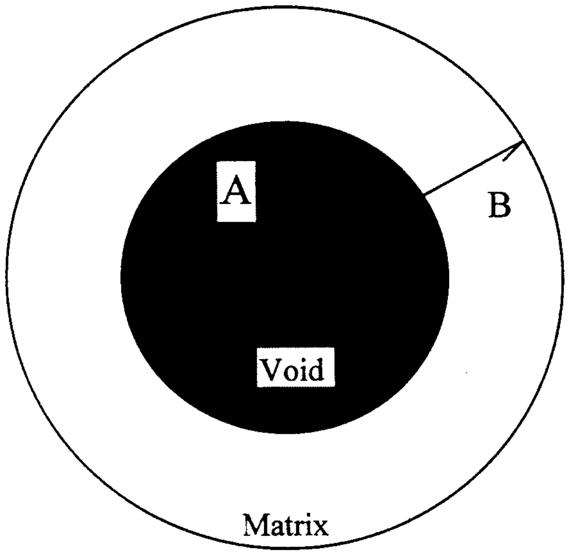

Figure 1 A diagrammatic representation of a porous, randomly faulted medium.

modulus of the unfilled nitrite rubber (G = 60 psi), and the experimental value of volumetric contraction ($\gamma = 0.23$), it was found that the **effective** Poisson's ratio is $v_{\text{eff}} = 0.492$ and the initial modulus of the bonded rubber disks in tension is $M_{\text{tens.}}$ = 2,990 psi. Using Warren's equation,[5] which correlates the **effective** Poisson's ratio v_{eff} with the volume fraction of growing voids, α, we found that the α grows from 0.2 to 2.2%.

In this article, the effluence of the porosity of the material on the *apparent* modulus, the strain field, and the stress field of a pancake sample subjected to triaxial stress was examined. Using the power balance equations, from a fracture mechanics point of view, one is able to extract mathematical equations for the strain, stress, and *apparent* modulus of the testing disks as a function of the volume fraction of growing flaw mechanisms inside the material. The proposed mathematical model involves two fitting parameters, the values of which can be determined using the experimental stress/strain curve received from the testing bonded elastomer disks.

POROSITY IN ELASTOMERIC MATERIALS

In this study, a damage variable, α, called the porosity, is defined as $\alpha = V_h/V$, where V_h denotes the volume of voids, and V is the volume of solids plus V_h. Following Jian and Ze-Ping,[6] the fracture criterion of the material is $\alpha \geq \alpha_{\text{crit}}$, where α_{crit} is the critical porosity of fracture. The porosity rate is composed of two parts, as follows:

$$\dot{\alpha} = \dot{\alpha}_n + \dot{\alpha}_g \tag{1}$$

where the "dot" denotes differential on time, $\dot{\alpha}_n$ is the contribution from the nucleation of voids, and $\dot{\alpha}_g$ is the contribution from the growth of existing voids. Nucleation and growth of voids can be considered in the failure analysis model. The major factors to influence microflaw nucleation are stress, strain, and temperature (see Curran et al.[7]). In this study, the porous material is assumed to be statistically homogeneous and isotropic so that it can be modeled by a homogeneous isotropic solid material. Along this theoretical analysis, the matrix is assumed to be *incompressible* and the spherical geometry is maintained during the growth of voids. A simplified model of porous element is assumed by considering a single spherical void of radius A in a sphere of radius B subjected to external stress $P(t)$ (see Fig. 1).

PROBLEM FORMULATION

The geometry and the coordinate system of the testing elastomer disks used for this analysis are shown in Figure 2. The radius and the thickness

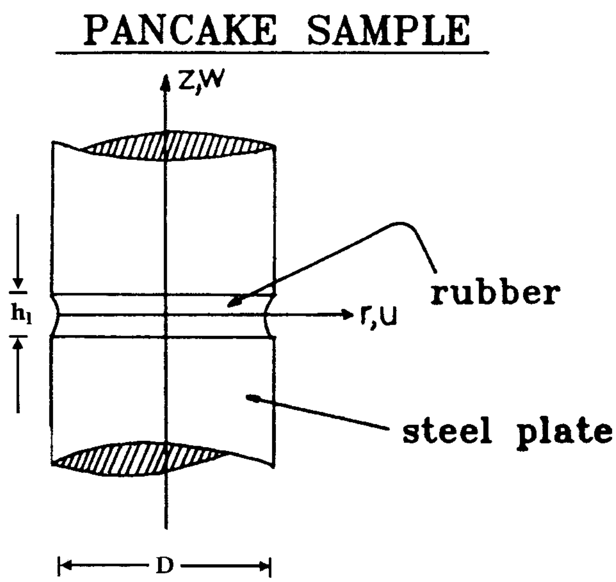

Figure 2 The geometry and the cylindrical coordinate system of the testing pancake. D, diameter of the cylindrical specimen; h_1, initial thickness of rubber; r,z denote the cylindrical coordinates of the specimen; u,w are the displacement fields in r,z directions, respectively.

of the sample were denoted by a and h, respectively. According to linear stress analysis, it can be shown that the longitudinal displacement $w(z)$ and the transverse displacement $u(r, z)$ are given by[3,4,8,9]

$$u(r, z) = AI_1(\rho)\left(1 - 4\frac{z^2}{h^2}\right) \quad (2)$$

and

$$w(z) = w(z) \quad (3)$$

where

$$\rho = \frac{2r}{h}\sqrt{\frac{3 - 6\nu}{2 - 2\nu}} \quad (4)$$

the $I_1(\rho)$ and v denote the modified Bessel function of first order[10] and Poisson ratio, respectively, and the coefficient A is given by

$$A = \frac{3\nu\alpha\varepsilon}{2[(1 - \nu)x\alpha I_0(x\alpha) - (1 - 2v)I_1(x\alpha)]} \quad (5)$$

where

$$x = \frac{2}{h}\sqrt{\frac{3 - 6\nu}{2 - 2\nu}} \quad (6)$$

For this system, the boundary conditions, due to incompressibility, are

$$u(r, z = h/2) = 0, w_z(z = h/2) = 0 \quad (7)$$

and at the upper boundary, the axial displacement is:

$$w(z = h/2) = \varepsilon h/2 \quad (8)$$

where ε denotes the strain in the pancake sample.

Using the average values for the stress field within the sample, one can prove[3,4] that the *effective* modulus for this testing system is given by

$$M = 2G\frac{\left((1-\nu) - \dfrac{2v^2}{(1-v)m - (1-2v)}\right)}{(1-2v)} \quad (9)$$

where

$$m = \theta\frac{I_0(\theta)}{I_1(\theta)}, \theta = \kappa\left(\frac{a}{h}\right), \kappa = \sqrt{\frac{3 - 6\nu}{2 - 2\nu}} \quad (10)$$

and G is the shear modulus of the material, which can be determined from the uniaxial tension data of the material. The *apparent modulus, M*, is defined as the ratio of the applied stress on the bonded disks divided by the strain within the disks. In previous studies, it was proved[3,4,11] that the *effective* Poisson's ratio v_{eff} is due to the presence of voids within the testing sample and is given by

$$\nu_{\text{eff}} = \frac{2 - 3\alpha}{4 - 3\alpha} \quad (11)$$

where α denotes the volume fraction of voids within the sample, that is,

$$\alpha = \frac{\frac{4}{3}\pi a^3_{\text{hole}} n_{\text{hole}}}{\pi a^2 h} \quad (12)$$

where a_{hole} and n_{hole} denote the radius and the number of voids in the elastomer disk, respectively.

Substitution of eq. (11) into eq. (9) yields

$$M^* \equiv \frac{M}{G} = \frac{4}{3}\left\{\frac{2m - 4 + 9\alpha - 9\alpha^2}{\alpha(2m - 3a)}\right\} \quad (13)$$

Also, the substitution of eq. (11) into eq. (10) yields

$$\theta = 3a^*\sqrt{\alpha} \quad (14)$$

where $a^* = a/h$ is the "aspect ratio" of the specimen.

A typical value of the aspect ratio for our experimental studies[1–5] was $a^* \approx 8$. It can be proved that for this value of the aspect ratio, the parameter m is close to θ,[2,3] that is,

$$m \approx \theta = 3a^*\sqrt{\alpha} \quad (15)$$

Substitution of eq. (15) into eq. (13) yields:

$$M^* = \frac{4}{3}\left\{\frac{6a^*\sqrt{\alpha} - 4 + 9\alpha - 9\alpha^2}{3\alpha\sqrt{\alpha}(2a^* - \sqrt{\alpha})}\right\} \quad (16)$$

Equation (16) yields the relationship between the normalized modulus of elasticity with the volume fraction of voids $\underline{\alpha}$.

The stress/strain law in a normalized form can be easily written as

$$T^* = M^*(\alpha)\varepsilon(\alpha) \qquad (17)$$

where M^* is given by eq. (16), the strain ε within the pancake sample is a function of void volume fraction α, and T^* denotes the normalized stress, that is, $T^* = T/G$. In the next section, a relation between the strain, ε, and the porosity, α, of the material is obtained using the concepts of fracture mechanics.

POWER BALANCE EQUATIONS

In this section, the power balance equations will be presented in order to evaluate the dependence of strain on the porosity of the material. Let P is the power input in the system due to external forces, U_1 is the potential energy of the system, U_2 is the kinetic energy, and U_3 is the surface energy; thus, the power balance equation is given by[12,13]

$$P = \dot{U}_1 + \dot{U}_2 + \dot{U}_3 \qquad (18a)$$

where

$$\begin{aligned} P &= 2T(\pi a^2)\dot{\omega} = T\dot{\varepsilon}V \quad (\omega = \varepsilon h/2) \\ U_1 &= \tfrac{1}{2}T\varepsilon V = \tfrac{1}{2}M\varepsilon^2 V \quad (V = \pi a^2 h) \\ \dot{U}_2 &= 0 \quad (\text{kinetic energy is neglected}) \\ U_3 &= \Gamma\alpha^n V \quad (\Gamma\alpha^n \text{ is empirical}) \end{aligned} \qquad (18b)$$

The dot above the righthand side of eq. (18) denotes the derivative with respect to time, and Γ is the surface energy per unit volume around the voids.

The substitution of eq. (18) into eq. (17) yields

$$\frac{P - \dot{U}_1}{V} = -\frac{\varepsilon^2}{2}\dot{M} = \Gamma n\alpha^{n-1}\dot{\alpha} \qquad (19)$$

where $\dot{\alpha}$ denotes the variation of void production in the sample with respect to testing time. The lefthand side of eq. (19) is know as the J integral. The parameters Γ and n must be determined by fitting with the experimental data. Hence, one is leading to a model with two fitting parameters.

Disregarding the algebraic details and using eq. (13), one can prove that

$$\dot{M}^* = -\frac{4\alpha^{\bullet}}{3\alpha^2}\left\{1 - \frac{4(2-3a)}{2m - 3\acute{a})} + \frac{(m^2-\theta^2)(2-3\alpha)^2}{(2m-3\alpha)^2}\right\} \qquad (20)$$

Substituting eq. (20) into eq. (19) yields

$$\varepsilon = \sqrt{\frac{3\Gamma n}{2G}}\,\frac{\alpha(n+1)/2}{\sqrt{\Phi(\alpha)}} \qquad (21)$$

where

$$\Phi(\alpha) = 1 - \frac{4(2-3a)}{2m-3\alpha)} + \frac{(m^2-\theta^2)(2-3\alpha)^2}{(2m-3\alpha)^2} \qquad (22)$$

Equation (21) relates the strain field ε in the testing sample with the porosity α of the material. For $m \approx \theta$, the last terms in eq. (22) drop out and one receives the following result:

$$\Phi(\alpha) = 1 - \frac{4(2-3a)}{(2m-3\alpha)} \qquad (23)$$

Since $\Phi(\alpha)$ must be positive and assuming the m is given by eq. (15), one obtains the following restriction for α:

$$3\sqrt{\alpha} \geq \sqrt{8 + (a^*)^2} - a^* \qquad (24)$$

For our experiments, $a^* = 8$, that is, $\sqrt{\alpha} \geq 0.16176$ or $\alpha \geq 0.02617 = \alpha_0$. Hence, in order for the theory to work, the volume fraction of voids must be greater than α_0. When $\alpha = \alpha_0$, then $\Phi = 0$, and from eq. (16), $M/G = 26.48$. Also, when $\alpha = 1$, then $\Phi = 1.09$ and $M/G = 1.304$. A plot of $\Phi(\alpha)$ as a function of the volume fraction of voids, α, is depicted in Figure 3. The normalized effective modulus of elasticity versus the void volume fraction is shown in Figure 4. In Figure 5, one can see the effluence of strain field on the void volume fraction for various values of fitting parameters, $3\Gamma n/2G$ and n.

The substitution of eqs. (16) and (21) into eq. (17) yields the stress/strain law, including the dependence on the void volume fraction, that is,

$$\frac{T}{G} = \frac{4}{3}\left\{\frac{2m - 4 + 9\alpha - 9\alpha^2}{\alpha(2m-3\alpha)}\right\} \times \left(\frac{3\Gamma}{2G}\right)^{1/2}\frac{\alpha^{(n+1)/2}}{\sqrt{\Phi}} \qquad (25)$$

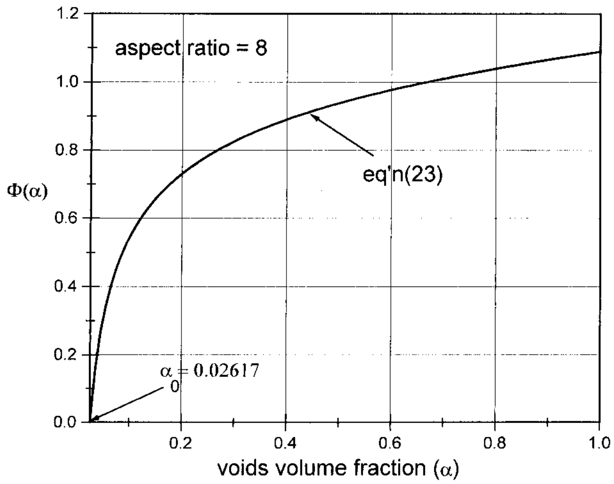

Figure 3 Plot of the function $\Phi(\alpha)$ versus the volume fraction of voids, α, based on eq. (23).

where m and Φ are defined by eqs. (15) and (23), respectively. Equation (25) is a new constitutive law for the pancake sample interconnecting the stress with the strain whenever the medium is no longer a pure elastic but is a porous randomly faulted medium.[14]

EXPERIMENTAL

In order to justify the proposed theory, a few experiments were run on a pancake sample with a radius of 3 inches and a thickness of 0.381 inches (that is, an aspect ratio of about 8). The material used in the experimental studies is described in Refs. 1–4. The geometry of the sample is shown in Figure 2, and the experimental stress versus strain curve is shown in Figure 6. The experiments were run on an MTS (machine for testing materials) with a strain rate of 0.026/min at room temperature.[11] An O-ring type of material was tested under uniaxial tension in an INSTRON testing machine (model 1026), and the shear modulus of the material was evaluated by using the Mooney-Rivlin plot. A value of $G = 68$ psi (or, Young's modulus = 204 psi) was estimated for the testing material. Notice that the elastomer is considered to be an incompressible rubber-like solid. It was previously shown[1–4] that the low modulus of the testing disks was due to the porosity of the material. It has been shown[1–4] that the porosity and/or flaws within the materials were developed during the curing process of the material.

NUMERICAL RESULTS

The experimental stress/strain curve of Figure 6 was fitted with eq. (25) in order to evaluate the fitting parameters $3\Gamma n/2G$ and n. One can take a low value of the porosity, α, say $\alpha_0 = 0.03 = 3\%$, and fit the experimental stress/strain curve using eq. (25). The fitting shows that the parameters $3\Gamma n/2G$ and n are equal to 0.1 and 1.5, respectively. Hence, the material parameters $3\Gamma n/2G$ and n can be easily determined by simple tension data of the testing specimen.

By numerical differentiation of the experimental stress/strain curve (Fig. 6), one can easily find the experimental normalized modulus of the testing material as a function of the strain within the solid under tension. The outcome of this procedure is shown in Figure 7. Using the proposed eq. (16) for the normalized modulus and the corresponding experimental curve for the modulus, one can determined the volume fraction of growing voids inside the testing sample. The values of α for various values of strain are shown in Figure 7. One can see that the void volume fraction increases as the modulus decreases and the strain increases. Since the values of fitting parameters are known, the strain field as a function of α is known using either eq. (21) or Figure 5. Therefore, one can easily estimate the growing of voids within a testing pancake sample using the experimental stress/strain curve in conjunction with the proposed theory.

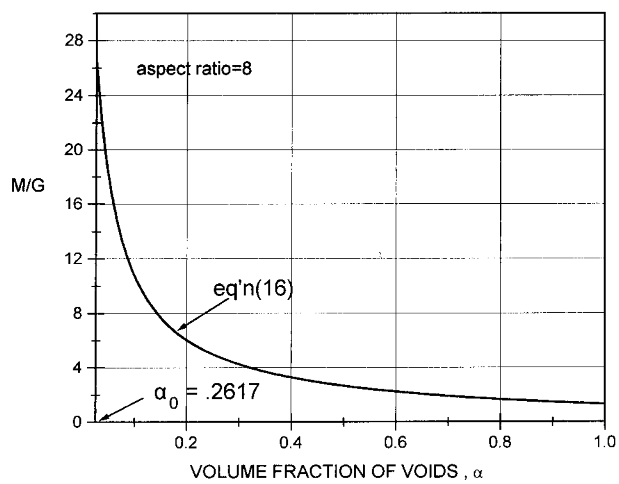

Figure 4 Plot of the normalized effective modulus $M^*(\alpha)$ versus the volume fraction of voids, α, based on eq. (16).

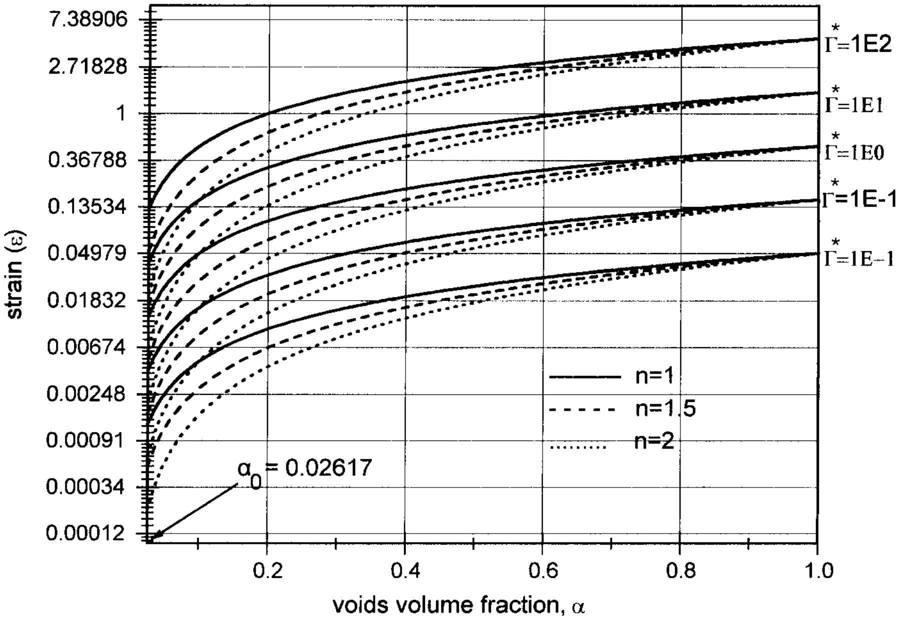

Figure 5 Plot of strain ε versus the volume fraction of voids, α, for various values of n and $3\Gamma n/2G$.

CONCLUSION

It was found that the strain field and the modulus of a pancake sample depend on the porosity of the material. As the strain increases inside the material, the volume fraction of voids increases and the modulus of elasticity decreases. Experimentally, this phenomenon was observed by myself and coworkers in previous studies[1,2] using the acoustic emission technique. The proposed mathe-

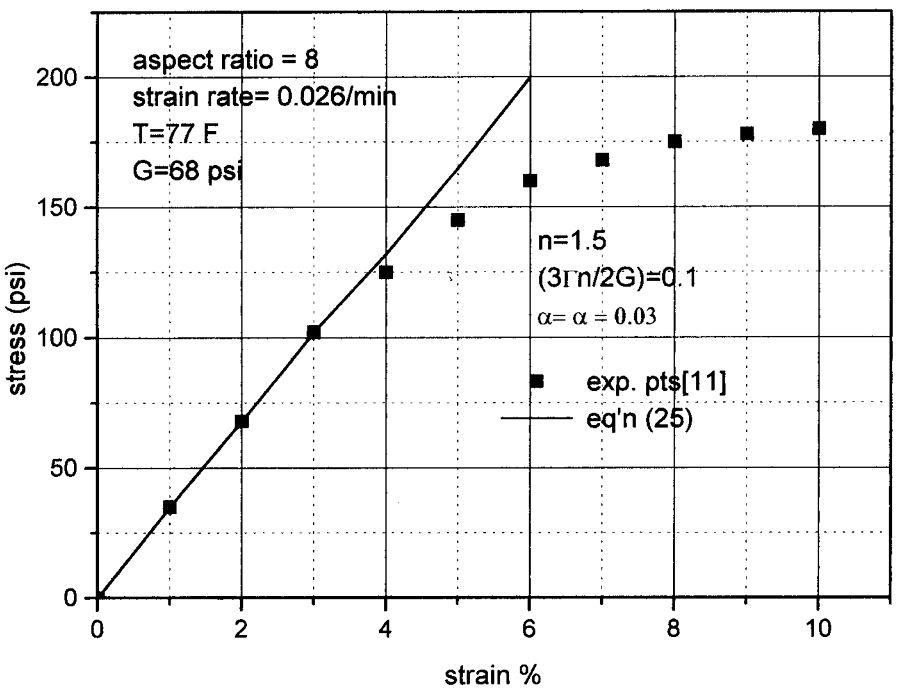

Figure 6 Experimental stress/strain curve of an unfilled nitrite rubber and fitting with eq. (25) in order to extract the values of n and $3\Gamma n/2G$.

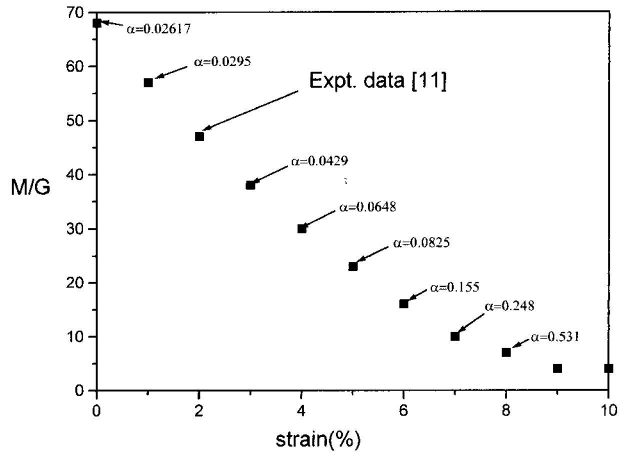

Figure 7 Experimental normalized modulus M/G versus the strain ε, with the corresponding values of volume fraction of voids, α, for given strain.

matical model involves two fitting parameters, the values of which can be determined using the experimental stress/strain curve from the testing bonded elastomer disks.

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